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Screening of charged singularities of random fields

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Abstract

Many types of point singularity have a topological index, or 'charge', associated with them. For example, the phase of a complex field depending on two variables can either increase or decrease on making a clockwise circuit around a simple zero, enabling the zeros to be assigned charges of ± 1. In random fields we can define a correlation function for the charge-weighted density of singularities. For many types of topologically charged singularity, this correlation function satisfies an identity which shows that the singularities 'screen' each other perfectly: a positive singularity is surrounded by an excess of concentration of negatives which exactly cancel its charge, and vice versa. This paper gives a simple and widely applicable derivation of this result. A counterexample where screening is incomplete is also exhibited.

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1. Introduction

This paper provides a simple and general explanation for a feature which has been noted in several types of point singularities of smooth randomly defined functions. The phenomenon is most easily described in terms of a specific example. Consider the set of zeros of a complex-valued differentiable random function ϕ, depending on two real variables $\mathbf{x} = (x, y)$. It is assumed that the statistical properties of $\phi(\mathbf{x})$ are translationally invariant. The phase of ϕ may either increase or decrease by 2π on traversing a clockwise circuit about a simple zero. Accordingly, the zeros may be described as carrying either positive or negative charges. It has been noted that positive zeros tend to be surrounded by negative ones, and vice versa: by analogy with models of ionic fluids and plasmas this has been described as a 'screening' effect. It is expressed quantitatively in terms of a correlation function C describing the charge-weighted density of zeros at positions $\mathbf{x}$ and $\mathbf{x} + \mathbf{X}$ (a precise definition will be given in

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section 2). Perfect screening is characterized by the relation

$$\int \mathrm{d}\mathbf{X}\, C(\mathbf{X}) = 0 \tag{1.1}$$

where the integral extends over the entire two-dimensional plane.

A screening relation of this type was first noted by Halperin [1] as a consequence of an analytical evaluation of such a correlation function in the case of zeros of complex functions. The screening relation was also discussed by Liu and Mazenko [2] in a related context. Later, the effect was surmised to exist in degeneracies of a random matrix model used to investigate the quantized Hall effect [3]. Numerical demonstrations of perfect screening for both the zeros and extrema of Gaussian random fields in two dimensions have been published [4]. More recently, analytical evaluation of correlation functions for several types of point singularity has provided further evidence that (1.1) is valid in many cases [5–7]. Previously, the existence of perfect screening has only been demonstrated by calculations of the correlation function $C(\mathbf{X})$ on a case-by-case basis.

The explanation given here (in section 2) is very simple and widely applicable. It depends upon an assumption that another correlation function, denoted by $D_0(s)$ in section 2, has an integral which is convergent. This assumption is verified (in the appendix) in a very general context for random fields with a Gaussian distribution. For non-Gaussian random fields it is expected to be much easier to establish convergence of this integral than to prove (1.1) by evaluating $C(\mathbf{X})$ exactly. The argument in section 2 uses the zeros of a complex function as an illustrative example, and section 3 discusses the more general applications. The approach is so general that one might expect that the perfect screening described by (1.1) may be present wherever there is a charge-neutral gas of point singularities, at least when the random fields do not have long-ranged correlations. One exception to (1.1), relating to components of eigenvectors of random matrices, is described and explained in section 4. This counterexample is of some physical interest because it arises in the topological description of the integer quantized Hall effect, discussed in [8–10].

This paper will use $\langle A \rangle$ to denote the ensemble average of a quantity A. Because of the assumption that the statistics are translationally invariant, the correlation function between $\phi(\mathbf{x})$ and $\phi(\mathbf{x}')$ is a function of $\mathbf{X} = \mathbf{x} - \mathbf{x}'$ only. The correlation function may therefore be written $\langle \phi(\mathbf{x})\phi(\mathbf{x}') \rangle = c(\mathbf{x} - \mathbf{x}')$. It is distinct from that occurring in (1.1), which describes the singularities of the field.

2. A derivation of the screening relation

Equation (1.1) will be derived for the case of singularities defined on the plane, using the zeros of a complex function of two real variables, $\phi(x, y)$, as an illustrative example.

The strategy is to consider the total topological charge Q enclosed by a curve $\mathcal{C}$ which surrounds an area $\mathcal{A}$. It will be sufficient to consider $\mathcal{C}$ to be a circle of radius R in what follows. The screening relation can be demonstrated in cases where Q can be obtained from a line integral:

$$Q = \int_{\mathcal{C}} \mathrm{d}s\, F(s) \tag{2.1}$$

where $\mathrm{d}s$ is an element of distance around the boundary and where $F(s)$ is a field that can be obtained from $\phi(x, y)$ by a 'local' procedure, so that F is a function of ϕ and its partial derivatives. In the case of zeros of a complex-valued function, Q is simply the phase change

on integrating around the boundary, so that

$$Q = \frac{1}{2\pi}\int_{\mathcal{C}} \mathrm{d}s\, \mathrm{Im}\left[\frac{1}{\phi}\frac{\mathrm{d}\phi}{\mathrm{d}s}\right]. \tag{2.2}$$

This expression is in the form (2.1), with $F = \mathrm{Im}[\phi'/\phi]/2\pi$.

Now let $\mathbf{x}_i$ be the locations of the singularities of the field ϕ. The total charge enclosed in $\mathcal{C}$ is

$$Q = \sum_{\{i\,|\,\mathbf{x}_i \in \mathcal{A}\}} q_i \tag{2.3}$$

where q_i are the charges of the individual zeros, which take the values ± 1. A charge-weighted density of zeros, $\rho(\mathbf{x})$, is defined:

$$\rho(\mathbf{x}) = \sum_i q_i \delta(\mathbf{x} - \mathbf{x}_i). \tag{2.4}$$

The total charge Q enclosed by $\mathcal{C}$ can also be expressed in terms of this density:

$$Q = \int_{\mathcal{A}} \mathrm{d}A\, \rho(\mathbf{x}). \tag{2.5}$$

Because it was assumed that the random process generating $\phi(\mathbf{x})$ is translationally invariant, correlation functions depend only upon the difference between coordinates. We consider random processes which are symmetric between positive and negative charges, so that $\langle\rho(\mathbf{x})\rangle = 0$. We define the mean density of zeros (without charge weighting) to be ρ_0. The correlation function that we consider, already referred to in (1.1), is defined by

$$C(\mathbf{X}) = \langle\rho(\mathbf{x}+\mathbf{X})\rho(\mathbf{x})\rangle. \tag{2.6}$$

This correlation function is the sum of a singular part, $\rho_0\delta(\mathbf{X})$, and a regular part, which according to (1.1) is expected to cancel the weight of the delta function when integrated over all $\mathbf{X}$.

Now consider the correlation function of the quantity $F(s)$ introduced in (2.1). Because the statistics of ϕ are translationally invariant and because $F(s)$ is obtained from ϕ by a local procedure, the correlation between $F(s)$ and $F(s')$ depends upon the distance d between the points labelled by s and s', and on the angle θ between the tangent vectors to the curve $\mathcal{C}$ at these points: we write

$$\langle F(s)F(s')\rangle = D(d,\theta). \tag{2.7}$$

The choice of the curve $\mathcal{C}$ is immaterial to the argument, and in the simple case where we take this to be a circle of radius R we have $d = 2R\sin(|s-s'|/2R)$, $\theta = s - s'/R$, so that

$$\begin{aligned}\langle F(s)F(s')\rangle &= D(|s-s'|,0) + O(1/R)\\ &= D_0(s-s') + O(1/R)\end{aligned} \tag{2.8}$$

where we define $D_0(d) = D(d,0)$, which is the limiting form of the correlation function when the tangent vectors at the points labelled by s and s' are parallel.

Using (2.1) and (2.7), $\langle Q^2\rangle$ is expressed in terms of the correlation function of F:

$$\langle Q^2\rangle = \int_{\mathcal{C}} \mathrm{d}s \int_{\mathcal{C}} \mathrm{d}s' D(d,\theta). \tag{2.9}$$

In order to prove (1.1) it will be assumed that the correlation function $D_0(s-s')$ decays faster than $1/|s-s'|$ as $|s-s'| \to \infty$, so that its integral is convergent. The appendix presents an argument showing that this assumption is justified if ϕ is a Gaussian random function and if the magnitude of the correlation $c(\mathbf{X}) = \langle\phi(\mathbf{X}+\mathbf{x})\phi(\mathbf{x})\rangle$ decreases sufficiently rapidly as

$|\mathbf{X}| \to \infty$. Using this assumption about $D_0(s)$, in the limit where $R \to \infty$, the integral (2.9) is dominated by the region where $s - s'$ is small:

$$\langle Q^2 \rangle = 2\pi R \int_{-\infty}^{\infty} \mathrm{d}s \, D_0(s) + O(1). \tag{2.10}$$

To gain information about the correlation function $C(\mathbf{X})$, consider the mean-squared charge: from (2.7) we have

$$\langle Q^2 \rangle = \int_{\mathcal{A}} \mathrm{d}\mathbf{x} \int_{\mathcal{A}} \mathrm{d}\mathbf{x}' \, C(\mathbf{x} - \mathbf{x}'). \tag{2.11}$$

Again, consider the case where $\mathcal{A}$ is the interior of a circle of radius R. If the integral on the left-hand side of (1.1) exists, then the integral in (2.11) is dominated by contributions from where $\mathbf{x}$ is close to $\mathbf{x}'$, when R is large, so that

$$\langle Q^2 \rangle = \pi R^2 \int \mathrm{d}\mathbf{X} \, C(\mathbf{X}) + O(R). \tag{2.12}$$

Equations (2.10) and (2.12) are only consistent if the term proportional to R^2 in (2.12) vanishes. This implies that (1.1) is satisfied.

If ϕ is not Gaussian, there does not appear to be any general argument which indicates that $D_0(s - s')$ decays sufficiently rapidly. The argument given in the appendix can be adapted to various types of non-Gaussian field, but different examples must be treated on a case-by-case basis. However, it is expected to be much easier to establish that the integral in (2.10) converges than to establish (1.1), which is an exact identity satisfied by a highly singular function of the underlying field ϕ.

3. Generalizations

The argument presented in section 2 may be extended to many different random fields in different dimensions: there is no requirement that it should be possible to determine the correlation function $C(\mathbf{X})$ exactly.

The argument for perfect screening might be applicable to any point singularities which carry indices which we term 'charges', when the gas of charges is, on average, neutral. The critical requirement is that the charge within a closed region can be determined from a surface integral involving a statistically stationary field F which was derived from the original random field ϕ by a locally defined procedure. It was assumed that this secondary field has a correlation function which decays sufficiently rapidly at infinity, and which depends only upon the separation of the pair of coordinates.

Most charged point singularities can be detected by using a surface integral. As a second example, consider stationary points of a real function of two real variables. These may be characterized by the Poincaré index, which is defined by taking a clockwise circuit around the singularity. The index is +1 if the angle of the gradient vector rotates by $+2\pi$ (i.e. in the same direction as the circuit), and -1 if the gradient vector rotates in the opposite direction to the circuit. Thus maxima and minima have index +1, and saddles have index -1. Perfect screening has also been demonstrated in this case by calculating the correlation function exactly [1, 2], but it is instructive to see how the argument of section 2 is adapted in this simple case.

If $\mathbf{x}(s)$ is the closed curve $\mathcal{C}$, and $\mathbf{v}(s) = \nabla\phi(\mathbf{x}(s))$ is the gradient vector at s, then the total Poincaré index for the stationary points within $\mathcal{C}$ is

$$Q = \frac{1}{2\pi} \int_{\mathcal{C}} \mathrm{d}s \, \frac{\mathbf{v} \wedge \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}}{|\mathbf{v}|^2} = \int_{\mathcal{C}} \mathrm{d}s \, F(s) \tag{3.1}$$

where the second equality defines $F(s)$ in this example. All of the arguments that were applied to the function $F(s)$ defined in (2.1) are equally valid for that defined in (3.1). It follows that the correlation function of the density of extrema weighted by their Poincaré index also exhibits perfect screening under quite general conditions.

The argument also extends directly to cases in higher dimensions, for example it can be used to explain the example of perfect screening which was discussed in [3]. The reasoning for this case will be summarized briefly: the reader should refer to the earlier papers cited here for definitions of the quantities. Reference [3] considered the density of degeneracies between pairs of levels in random Hermitian matrices, which were a periodic function of three parameters, x_1, x_2 and x_3. There is a topological charge, the Chern index, associated with the energy levels of a two-parameter family of Hermitian matrices which are periodic in two parameters, x_1 and x_2. The Chern index is an integer which may represent a quantized Hall conductance associated with each energy level [8, 9]. Varying the third parameter x_3 allows pairs of levels to become degenerate. When two levels become degenerate, the Chern index on one level increases by one, while that of the other level decreases by one. This enables the degeneracies to be assigned a charge of ± 1, depending upon the sign of the change of the Chern index of the upper level resulting from increasing x_3 [10]. Reference [10] also shows that the total charge of the degeneracies of a given level (with label n, say) within a closed region of the three parameter space is equal to the integral of the Berry phase 2-form V_n [11] over the surface of the region.

The random matrix model discussed in [3] has statistical properties which are translationally invariant in $\mathbf{x} = (x_1, x_2, x_3)$ space, and this symmtetry means that degeneracies are equally likely to have either sign. This is analogous to the situation described in section 2: we have a homogeneous distribution of 'particles' (degeneracies) which are equally likely to have positive and negative charges, and the total charge Q within a three-dimensional region is obtained by integrating a function, the Berry phase 2-form, over its surface. Considering a spherical volume of radius R, and writing $\langle Q^2 \rangle$ in terms of the correlation function of charge density, gives an expression analogous to (2.12), with the leading term being $4\pi R^3/3$ multiplied by the integral of the charge density correlation function. Expressing $\langle Q^2 \rangle$ in terms of the surface integral leads to an expression analogous to (2.10) in which the leading term is $4\pi R^2$ multiplied by an integral of the correlation function of the 2-form, $C(\mathbf{x}-\mathbf{x}') = \langle V_n(\mathbf{x}) V_n(\mathbf{x}') \rangle$. Provided this correlation function vanishes faster than $1/|\mathbf{x}-\mathbf{x}'|^2$, that integral converges, and the correlation function of the charge density satisfies (1.1) (with the integral now being evaluated over a three-dimensional space). This argument explains the screening relation noted in [3].

4. A counterexample: zeros of eigenvector components

The discussion in the previous section indicates that the argument explaining perfect screening is very general, and it might be suspected that perfect screening is universal in charge-neutral gases of singularities. However this section will describe a counterexample, which arises from the same random matrix model as was considered in [3]. The mathematical structure of the model will be explained, but the reader should refer to [3] for a discussion of the physical motivation of this model.

Consider a complex Hermitian $N \times N$ matrix $\tilde{H}$, with elements $H_{nm}(x_1, x_2)$ which are functions of two real parameters, x_1 and x_2. The matrix elements H_{nm} can be Gaussian random functions, depending smoothly on $\mathbf{x} = (x_1, x_2)$, with a correlation function between $H_{nm}(\mathbf{x})$ and $H_{n'm'}(\mathbf{x}')$ which depends only upon $\mathbf{x} - \mathbf{x}'$ and which decays faster than $1/|\mathbf{x} - \mathbf{x}'|$ as $|\mathbf{x} - \mathbf{x}'| \to \infty$. In this case each component of any eigenvector of $\tilde{H}$ is a complex random

function of x_1 and x_2. We consider one arbitrary eigenvector component which will be termed $\phi(x_1, x_2)$. The function $\phi(x_1, x_2)$ is differentiable at (x_1, x_2) unless the corresponding eigenvalue is degenerate at (x_1, x_2). Three parameters must be varied to cause degeneracies of eigenvalues of a complex Hermitian matrix, so that we expect $\phi(x_1, x_2)$ to be differentiable everywhere in the (x_1, x_2) plane.

Let us consider the zeros of ϕ. There is nothing to favour one index of these zeros over the other, so that the distribution of zeros across the (x_1, x_2) plane is charge neutral. We can ask whether the screening relation applies to the zeros of the eigenvector component $\phi(x_1, x_2)$. It will be demonstrated that in the case of 2×2 matrices screeening the screening is not perfect. It will be argued that screening is also not perfect when $N > 2$.

The indices of zeros of a component of an eigenvector of a two-parameter family of Hermitian matrices play a role in the topological characterization of the integer quantized Hall effect [9]. In the case of independent electrons moving through a perfect two-dimensional crystal, the electron states form bands labelled by two Bloch wavevectors, x_1 and x_2. The Hamiltonian is a periodic function of x_1 and x_2, with a unit cell which is termed the Brillouin zone. The Hall conductance of a band is $Q\,e^2/h$, where Q is an integer. One way to calculate Q involves looking at any component ϕ of the eigenvector defining the wavefunction of the band. All of the zeros of ϕ within a Brillouin zone are located, and their indices $q_i = \pm 1$ are determined. The Hall conductance integer Q is then the sum of the indices: $Q = \sum_i q_i$.

In cases where the electrons move in a simple periodic potential the integers Q can be large (although these situations would be very hard to probe experimentally). If the system is disordered, or the unit cell is large, it is reasonable to propose using a random matrix model for the statistical properties of the Chern numbers. An appropriate random matrix model is described in [3]. The mean Chern number must (by symmetry) be equal to zero: $\langle Q \rangle = 0$. The simplest statistical characterization of the Chern numbers is through their variance $\langle Q^2 \rangle$.

The random matrix model in [3] was investigated by a combination of analytical and numerical approaches. Numerical experiments reported there support an expression of the form

$$\langle Q^2 \rangle = \alpha \mathcal{A} \sigma^2 \nu^2 \tag{4.1}$$

where

$$\sigma^2 = \det\left[\left\langle \frac{\partial E_n}{\partial x_i}\,\frac{\partial E_n}{\partial x_j}\right\rangle\right] \tag{4.2}$$

is a measure of the sensitivity of energy levels to perturbation, ν is the density of states per unit area, $\mathcal{A}$ is the area of the Brillouin zone and α is a constant, determined numerically to be approximately 0.2 in the limit where the dimension of the matrix is large. Because equation (4.1) is proportional to $\mathcal{A}$, perfect screening does not apply in the case of zeros of eigenvector components. We have already seen that screening is predicted by an argument with a very broad range of applicability. It is desirable to understand why screening fails at two levels. Firstly, which of the criteria stated in the derivation are not met? And secondly, can it be seen by an explicit calculation that screening is absent?

First, let us consider the reason why the demonstration presented in section 2 cannot be applied. The argument uses a function $F(s)$ which yields the phase change upon integration around the boundary, and the function which was defined there, $F(s) = (1/\phi)\mathrm{Im}(\mathrm{d}\phi/\mathrm{d}s)/2\pi$, is also appropriate in this problem. It was assumed that the function $F(s)$ is differentiable and has a correlation function which is statistically stationary (such that on the circular boundary $\langle F(s)F(s')\rangle$ is a function of $s - s'$ only), and that this correlation function decays faster than $1/|s - s'|$. If ϕ is a component of an eigenvector of a random matrix, these assumptions about $F(s)$ may be challenged. In order to apply (2.1) we must asume that $\phi(x_1, x_2)$ is a continuous

function of x_1 and x_2. The eigenvectors of a matrix may be multiplied by any complex number of modulus unity, $\exp[\mathrm{i}\theta(x_1, x_2)]$ and remain eigenvectors. When constructing $\phi(x_1, x_2)$ from the random matrix $\tilde{H}(x_1, x_2)$, the eigenvectors can be produced by some fixed algorithm (for matrices of dimension $N > 4$, this is necessarily numerical). This results in a 'raw' form for $\phi(x_1, x_2)$ which is both statistically stationary and locally correlated. However, this function will have discontinuities in phase where the diagonalization algorithm has branches, which must be removed by multiplying by a suitable factor $\exp[\mathrm{i}\theta(x_1, x_2)]$. The function $\theta(x_1, x_2)$ cannot be constructed by any locally defined algorithm. It is therefore possible that the resulting regularized $\phi(x_1, x_2)$ may be non-stationary, or non-locally correlated, or both.

This general argument is a little unsatisfying, because it does not show that the perfect screening effect must fail. However the source of the failure can be seen clearly in the case of a 2×2 Hermitian random matrix, with elements

$$\tilde{H}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) & f_2(\mathbf{x}) + \mathrm{i} f_3(\mathbf{x}) \\ f_2(\mathbf{x}) - \mathrm{i} f_3(\mathbf{x}) & -f_1(\mathbf{x}) \end{pmatrix} \tag{4.3}$$

where $\mathbf{x} = (x_1, x_2)$, and we assume that the real-valued random functions $f_i(\mathbf{x})$ satisfy

$$\langle f_i(\mathbf{x}) \rangle = 0 \tag{4.4}$$

$$\langle f_i(\mathbf{x} + \mathbf{x}_0) f_j(\mathbf{x}_0) \rangle = \delta_{ij} C_i(|\mathbf{x}|). \tag{4.5}$$

The eigenvalues $E_\pm$ and corresponding normalized eigenvectors $\mathbf{u}_\pm$ are

$$E_\pm = \pm\sqrt{f_1^2 + f_2^2 + f_3^2} \tag{4.6}$$

$$\mathbf{u}_\pm = \frac{\exp[\mathrm{i}\theta_\pm]}{\sqrt{(E_\pm - f_1)^2 + f_2^2 + f_3^2}} \begin{pmatrix} f_2 + \mathrm{i} f_3 \\ E_\pm - f_1 \end{pmatrix} \tag{4.7}$$

where all the variables are real functions of $\mathbf{x}$, and where $\theta_\pm(\mathbf{x})$ is chosen so that the first component of $\mathbf{u}_\pm$ is a continuous function of $\mathbf{x}$. Let us consider the first component of the eigenvector corresponding to the E_+ branch, calling this function $\phi(\mathbf{x})$. Writing $\psi(\mathbf{x}) = f_2(\mathbf{x}) + \mathrm{i} f_3(\mathbf{x})$ and dropping the subscript on θ_+, we have

$$\phi(\mathbf{x}) = \frac{\exp[\mathrm{i}\theta]\psi}{\sqrt{\left[\sqrt{f_1^2 + |\psi|^2} - f_1\right]^2 + |\psi|^2}}. \tag{4.8}$$

Consider the behaviour of ϕ at a zero of ψ. At each zero of ψ, we have $E_+ = |f_1|$. We must consider two cases, depending upon whether f_1 is positive or negative. If f_1 is negative at the zero of ψ, then in the neighbourhood of this zero we have

$$\phi \sim \frac{\exp[\mathrm{i}\theta]\psi}{2|f_1|} \tag{4.9}$$

so that ϕ has a zero with the same index as ψ, θ having no singularity at these points. In the case where f_1 is positive at the zero of ψ, in the neighbourhood of this point we have

$$\phi \sim \frac{\exp[\mathrm{i}\theta]\psi}{|\psi|}. \tag{4.10}$$

At these zeros of ψ, ϕ does not have a zero. If ϕ is to be a continuous function, θ must have a singularity in which it increments by $\pm 2\pi$ on making a circuit around the zero of ψ, in order to cancel the phase singularity of $\psi/|\psi|$.

We can now give a clear picture of why screening is not perfect in the eigenvector component ϕ, for the special case of this 2×2 random matrix model. We have seen that the zeros of ϕ have the same indices as those of ψ, which do exhibit perfect screening. However, only a randomly chosen half of the zeros of ψ (selected by the criterion that f_1 is negative at the zero) are represented as zeros of ϕ. It is therefore not expected that the delicate balance implied by perfect screening will be present in the zeros of ϕ. In the limit where the correlation length of $f_1(\mathbf{x})$ is made short compared to that of f_2 and f_3, the deleted zeros are selected completely randomly.

This conclusion can be expressed quantitatively as follows. The sign-weighted density of zeros of the eigenvector component $\phi(\mathbf{x})$ is

$$\rho_\phi(\mathbf{x}) = \chi(f_1(\mathbf{x}))\rho_\psi(\mathbf{x}) \tag{4.11}$$

where $\rho_\psi(\mathbf{x})$ is the sign-weighted density of zeros of $\psi = f_2 + \mathrm{i}f_3$, and the factor $\chi(f_1(\mathbf{x}))$ selects those zeros for which $f_1(\mathbf{x})$ is negative: $\chi(x)$ is unity if x is negative, zero otherwise. Using the fact that f_1 is independent of ψ, we have

$$\begin{aligned} C_\phi(\mathbf{x}) &= \langle \chi(f_1(\mathbf{x}_0))\chi(f_1(\mathbf{x}_0+\mathbf{x}))\rangle C_\psi(\mathbf{x}) \\ &= C_\chi(\mathbf{x})C_\psi(\mathbf{x}) \end{aligned} \tag{4.12}$$

where the second equality defines $C_\chi(\mathbf{x})$. The function C_ψ satisfies equation (1.1). The function C_ϕ need not. For example, if f_1 has a correlation length which is short compared to f_2 and f_3, the correlation length of C_χ will also be short compared to that of C_ψ. Equation (4.12) shows that in this limiting case the screening effect would be absent from the eigenvector component ϕ.

Appendix

This appendix discusses the calculation of the correlation function $\langle F(s)F(s')\rangle$, in the limit where the separation $|s-s'|$ is large. It is assumed that $F(s)$ is a function of the field $\phi(s)$ and its derivative $\phi'(s)$, that is $F(s) = \mathcal{F}(\phi(s), \phi'(s))$ for some function $\mathcal{F}(a, b)$, but the argument is easily extended to cases where $F(s)$ depends on higher derivatives. The argument can also be adapted to higher dimensions. The calculation discussed here does use the assumption that the field ϕ is Gaussian, but extensions to non-Gaussian fields are possible.

It will be shown that $\langle F(s)F(s')\rangle$ decays no more slowly (as $|s-s'| \to \infty$) than the most slowly decaying of the correlations $\langle\phi(s)\phi(s')\rangle$, $\langle\phi'(s)\phi'(s')\rangle$, $\langle\phi'(s)\phi(s')\rangle$. The correlation functions of derivatives are related to the derivatives of the correlation function, for example if $\langle\phi(s)\phi(s')\rangle = c(s-s')$, then $\langle\phi'(s)\phi'(s')\rangle = -c''(s-s')$.

The joint probability density for N Gaussian random variables $(x_1, x_2, \ldots, x_N)^T = \mathbf{X}$, all of which satisfy $\langle x_i\rangle = 0$, is

$$\mu(\mathbf{X}) = [(2\pi)^N \det\tilde{C}]^{-1/2} \exp\left(-\tfrac{1}{2}\mathbf{X}^T\tilde{C}^{-1}\mathbf{X}\right) \tag{A.1}$$

where $\tilde{C}$ is the correlation matrix of the random variables x_i, with elements

$$C_{ij} = \langle x_i x_j\rangle. \tag{A.2}$$

Now write $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2)^T$, $\mathbf{x}_1 = (\phi(s), \phi'(s))^T$, $\mathbf{x}_2 = (\phi(s'), \phi'(s'))^T$, and write the correlation matrix (A.2) in the form

$$\tilde{C} = \begin{pmatrix} \tilde{A} & \epsilon(s-s')\tilde{a}(s-s') \\ \epsilon(s-s')\tilde{a}(s-s') & \tilde{A} \end{pmatrix} \tag{A.3}$$

where $\tilde{A}$ and $\tilde{a}$ are 2×2 matrices; $\tilde{A}$ is independent of $s-s'$, $\tilde{a}$ has elements which are bounded as $|s-s'| \to \infty$ and ϵ is a function of $s-s'$ which decays no more rapidly than the most slowly decreasing correlation function of ϕ and ϕ' as $|s-s'| \to \infty$.

When $|\epsilon|$ is sufficiently small, the inverse of $\tilde{C}$ is approximated by

$$\tilde{C}^{-1} = \begin{pmatrix} \tilde{A}^{-1} & -\epsilon\tilde{A}^{-1}\tilde{a}\tilde{A}^{-1} \\ -\epsilon\tilde{A}^{-1}\tilde{a}\tilde{A}^{-1} & \tilde{A}^{-1} \end{pmatrix} + O(\epsilon^2). \tag{A.4}$$

We also have $\det(\tilde{C}) = [\det(\tilde{A})]^2 + O(\epsilon^2)$. Using (A.4) and (A.1), the joint probability density for $\mathbf{x}_1$ and $\mathbf{x}_2$ is

$$\begin{aligned} \mu(\mathbf{X}) = \mu(\mathbf{x}_1, \mathbf{x}_2) &= \frac{1}{(2\pi)^2 \det(\tilde{A})} \exp\left[-\frac{1}{2}(\mathbf{x}_1, \mathbf{x}_2)^T \tilde{C}^{-1}(\mathbf{x}_1, \mathbf{x}_2)\right] + O(\epsilon^2) \\ &= \mu_0(\mathbf{x}_1)\mu_0(\mathbf{x}_2) \exp\left[\epsilon \mathbf{x}_1^T \tilde{A}^{-1}\tilde{a}\tilde{A}^{-1}\mathbf{x}_2\right] + O(\epsilon^2) \end{aligned} \tag{A.5}$$

where

$$\mu_0(\mathbf{x}) = \frac{1}{2\pi\sqrt{\det\tilde{A}}} \exp\left[-\frac{1}{2}\mathbf{x}^T \tilde{A}^{-1}\mathbf{x}\right] \tag{A.6}$$

is the marginal probability density of $\mathbf{x} = (\phi, \phi')$.

The correlation function $\langle F(s)F(s')\rangle = \langle \mathcal{F}(\mathbf{x}_1)\mathcal{F}(\mathbf{x}_2)\rangle$ is

$$\langle F(s)F(s')\rangle = \int d\mathbf{x}_1 \int d\mathbf{x}_2\, \mu(\mathbf{x}_1, \mathbf{x}_2)\mathcal{F}(\mathbf{x}_1)\mathcal{F}(\mathbf{x}_2). \tag{A.7}$$

The expectation $\langle F(s)\rangle$ is given by an analogous expression, in which $\mathcal{F}(\mathbf{x})$ is integrated over $\mathbf{x}$ with weight $\mu_0(\mathbf{x})$. Expanding the exponential in (A.5), and using the fact that $\langle F(s)\rangle = 0$ gives

$$\begin{aligned} \langle F(s)F(s')\rangle &= \epsilon \int d\mathbf{x}_1 \int d\mathbf{x}_2\, \mu_0(\mathbf{x}_1)\mu_0(\mathbf{x}_2)\mathcal{F}(\mathbf{x}_1)\mathcal{F}(\mathbf{x}_2)\mathbf{x}_1^T \tilde{A}^{-1}\tilde{a}\tilde{A}^{-1}\mathbf{x}_2 + O(\epsilon^2) \\ &= \epsilon \mathbf{g}^T \tilde{A}^{-1}\tilde{a}\tilde{A}^{-1}\mathbf{g} + O(\epsilon^2) \end{aligned} \tag{A.8}$$

where

$$\mathbf{g} = \int d\mathbf{x}\, \mathbf{x}\, \mathcal{F}(\mathbf{x})\mu_0(\mathbf{x}). \tag{A.9}$$

Equation (A.8) shows that the correlation function $\langle F(s)F(s')\rangle$ is $O(\epsilon)$, implying that it is bounded by a multiple of the most slowly decaying correlation function of the fields occurring as arguments of the function $\mathcal{F}$. It follows that the correlation function of $F(s)$ will decay sufficiently rapidly at infinity if the correlation function of the fields has a sufficiently rapid decay.

This approach extends directly to non-Gaussian fields in the case where the joint probability density factorizes in the limit $|s - s'| \to \infty$, such that $\mu(\mathbf{x}, \mathbf{x}') = \mu_0(\mathbf{x})\mu_0(\mathbf{x}')$ $[1 + O(\epsilon)]$.

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